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Ramanujan complexes and high dimensional expanders*

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Abstract. Expander graphs in general, and Ramanujan graphs in particular, have been of great interest in the last four decades with many applications in computer science, combinatorics and even pure mathematics. In these notes we describe various efforts made in recent years to generalize these notions from graphs to higher dimensional simplicial complexes.

Keywords and phrases: expanding graphs, high dimensional expanders, Ramanujan graphs, Ramanujan complexes, overlapping property

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Contents

0. Introduction 137
1. Ramanujan graphs 138
2. Ramanujan complexes 147
3. High dimensional expanders 156

0. Introduction

Expander graphs are highly connected finite sparse graphs. These graphs play a fundamental role in computer science and combinatorics (*cf.* [Lub94, HLW06],

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and the references within) and in recent years even found numerous applications in pure mathematics ([Lub12]). Among these graphs, Ramanujan graphs stand out as optimal expanders (at least from the spectral point of view). The theory of expanders and Ramanujan graphs has led to a very fruitful interaction between mathematics and computer science (and between mathematicians and computer scientists). In the early days, deep mathematics (e.g. Kazhdan property (T) and the Ramanujan conjecture) has been used to construct expanders and Ramanujan graphs. But recently, the theory of computer science pays its debt to mathematics and expanders start to appear more and more also within pure mathematics.

The fruitfulness of this theory calls for a generalization to high dimensional theory. Here the theory is much less developed. The goal of these notes is to describe some of these efforts and to call the attention of the mathematical and computer science communities to this challenge. We strongly believe that a beautiful and useful theory is waiting for us to be explored.

Most of the notes will be dedicated to the story of Ramanujan complexes. These generalizations of Ramanujan graphs, which has been developed in [CSŻ03, Li04, LSV05a, LSV05b, Sar07] became possible by the significant development of the theory of automorphic forms in positive characteristic and especially the work of L. Lafforgue [Laf02]. In Chap. 1, we will describe the classical theory of Ramanujan graphs, in a way which will pave the way for a smooth presentation in Chap. 2, of the much more complicated theory of Ramanujan complexes.

The situation with high dimensional expanders is more chaotic. Here it is not even agreed what should be the "right" definition. Several generalizations of the concept of expander graph have been suggested, which are not equivalent. It is not clear at this point which one is more useful. Each has its own charm and part of the active research on this subject is to understand the relationships between the various definitions.

We describe these activities briefly in Chap. 3. It can be expected (and, in fact, I hope!) that these notes will not be up to date by the time they will appear in press. . . .

1. Ramanujan graphs

In this chapter we will survey Ramanujan graphs, which are optimal expanding graphs from a spectral point of view. The material is quite well known by now and has been described in various places ([LPS88, Sar90, Lub94, Val97]). We present it here in a way which will pave the way for the high dimensional generalization—the Ramanujan complexes—which will come in the next chapter.

1.1. Eigenvalues and expanders

Let $X = (V, E)$ be a finite connected k-regular graph, $k \geq 3$, with a set V of n vertices, and adjacency matrix $A = A_X$, i.e., A is an $n \times n$ matrix indexed by the vertices of X and A_{ij} is equal to the number of edges between i and j (which is either 0 or 1 if X is a simple graph).

Definition 1.1.1. *The graph X is called* Ramanujan *if for every eigenvalue λ of the symmetric matrix A, either $\lambda = \pm k$ ("the trivial eigenvalues") or $|\lambda| \leq 2\sqrt{k-1}$.*

Recall that k is always an eigenvalue of A (with the constant vector/function as an eigenfunction) while $-k$ is an eigenvalue of A if and only if X is bi-partite, i.e., the vertices of X can be divided into two disjoint sets Y and Z and every edge e in E, has one endpoint in Y and one in Z. In this case, the eigenfunction is 1 on Y and -1 on Z.

Ramanujan graphs have been defined and constructed in [LPS88] (see also [Mar88] and see [Sar90, Lub94, Val97] for a more comprehensive treatment). The importance of the number $2\sqrt{k-1}$ comes from the Alon–Boppana Theorem which asserts that for any fixed k, no better bound can be obtained on the non-trivial eigenvalues of an infinite sequence of finite k-regular graphs.

Theorem 1.1.2 (Alon–Boppana (*cf.* [LPS88, Nil91])). *For a finite connected k-regular graph X, denote*

$$\mu_1(X) = \max\{\lambda \mid \lambda \text{ an eigenvalue of } A \text{ and } \lambda \neq k\},$$
$$\mu_0(X) = \max\{|\lambda| \mid \lambda \text{ an eigenvalue of } A \text{ and } \lambda \neq k\},$$
$$\mu(X) = \max\{|\lambda| \mid \lambda \text{ an eigenvalue of } A \text{ and } \lambda \neq \pm k\}.$$

If $\{X_i\}_{i=1}^{\infty}$ is a sequence of such graphs with $|X_i| \to \infty$ (where $|X_i|$ is the size of X_i), then

$$\liminf_{i \to \infty} \mu(X_i) \geq 2\sqrt{k-1}.$$

The hidden reason for the number $2\sqrt{k-1}$ is: All the finite connected k-regular graphs are covered by the k-regular tree, $T = T_k$. Let A_T be the adjacency operator of T, i.e., for every function f on the vertices of T and for every vertex x of it,

$$A_T(f)(x) = \sum_{y \sim x} f(y)$$

namely, A_T sums f over the neighbors of x. Then A_T defines a self adjoint operator $L^2(T) \to L^2(T)$.

Proposition 1.1.3 ([Kes59]). *The spectrum of A_T is* $[-2\sqrt{k-1}, 2\sqrt{k-1}]$.

Of course, k is not an eigenvalue of A_T as the constant function is not in L^2. It is even not in the spectrum (unless $k = 2$, in which case T_k is a Cayley graph of the amenable group $\mathbb{Z}$, but this is a different story). But, k is necessarily an eigenvalue for all the adjacency operators induced on the finite quotients $\Gamma\backslash T$, where Γ is a discrete cocompact subgroup of $\mathrm{Aut}(T)$. Similarly, $-k$ is an eigenvalue of the finite quotient $\Gamma\backslash T$ if it is bi-partite (which happens if $\Gamma = \pi_1(\Gamma\backslash T)$ preserves the two-coloring of the vertices of T). Now, Ramanujan graphs are the "ideal objects" having their non-trivial spectrum as good as the "ideal object" T.

There is another way to characterize Ramanujan graphs. These are the graphs which satisfy the "Riemann hypothesis", i.e., all the poles of the Ihara zeta function associated with the graph lie on the line $\Re(s) = \frac{1}{2}$. See [Lub94, §4.5] and especially the works of Ihara [Iha66], Sunada [Sun88] and Hashimoto [Has89].

The work of Ihara showed the close connection between number theoretic questions and the combinatorics of some associated graphs. While it was Satake [Sat66] who showed how the classical Ramanujan conjecture can be expressed in a representation theoretic way. These works have paved the way to the explicit constructions of Ramanujan graphs to be presented in §1.2 and §1.3.

Ramanujan graphs have found numerous applications in combinatorics, computer science and pure mathematics. We will not describe these but rather refer the interested readers to the thousands references appearing in google scholar when one looks for Ramanujan graphs.

We should mention however their main application and original motivation: expanders.

Definition 1.1.4. *For X a k-regular graph on n vertices, denote:*

$$h(X) = \min_{0<|A|<|V|} \frac{n \cdot |E(A, V\backslash A)|}{|A||V\backslash A|}$$

where $E(A, B)$ is the set of edges from A to B. We call $h(X)$ the Cheeger constant of X.

Remark 1.1.5. In most references, the Cheeger constant is defined as

$$\overline{h}(X) = \min_{0<|A|\leq\frac{|V|}{2}} \frac{|E(A, V\backslash A)|}{|A|}.$$

Clearly $\overline{h}(X) \leq h(X) \leq 2\overline{h}(X)$. For our later purpose, it will be more convenient to work with $h(X)$.

Definition 1.1.6. *The graph X is called an ε-expander (for $0 < \varepsilon \in \mathbb{R}$) if $h(X) \geq \varepsilon$.*

Expander graphs are of great importance in computer science. Ramanujan graphs give outstanding expanders due to the following result:

Theorem 1.1.7 ([Tan84, Dod84, AM85, Alo86]). *For X as above,*

$$\frac{h^2(X)}{8k} \leq k - \mu_1(X) \leq h(X).$$

In particular, Ramanujan k-regular graphs are ε-expanders with $\varepsilon = k - 2\sqrt{k-1}$ (or if one prefers the more standard notation $\overline{h}(X) \geq \frac{k}{2} - \sqrt{k-1}$).

A very useful result in many applications is the following Expander Mixing Lemma:

Proposition 1.1.8. *For $X = (V, E)$ as above and for every two subsets A and B of V,*

$$\left| E(A, B) - \frac{k|A||B|}{|V|} \right| \leq \mu_0(X)\sqrt{|A||B|}.$$

Note that $\frac{k|A||B|}{|V|}$ is the expected number of edges between A and B if X would be a "random k-regular graph". So, if $\mu_0(X)$ is small, e.g. if X is Ramanujan, it mimics various properties of random graphs. This is one of the characteristics which make them so useful.

There is no easy method to construct Ramanujan graphs. Let us better be more precise here: There are many ways to get, for a fixed k, finitely many k-regular Ramanujan graphs (see [Lub94, Chapter 8]), but there is essentially only one known explicit way to get, for a fixed k, infinitely many k-regular Ramanujan graphs. This will be described in the next section. A recent breakthrough [MSS13] showed, in a non-explicit way, that for every $k \geq 3$, there are infinitely many bi-partite k-regular Ramanujan graphs.

In the next subsection we will describe the Bruhat–Tits tree and present the basic theory that will enable us in the following subsection to get explicit constructions of Ramanujan graphs.

1.2. Bruhat–Tits trees and representation theory of PGL_2

Let F be a local field (e.g. $F = \mathbb{Q}_p$ the field of p-adic numbers, or a finite extension of it, or $F = \mathbb{F}_q((t))$ the field of Laurent power series over the finite field $\mathbb{F}_q$) with ring of integers $\mathcal{O}$ (e.g. $\mathcal{O} = \mathbb{Z}_p$ or $\mathcal{O} = \mathbb{F}_q[[t]]$), maximal ideal $\mathfrak{m} = \pi\mathcal{O}$ where π is a fixed uniformizer, i.e., an element of $\mathcal{O}$ with valuation $\nu(\pi) = 1$ (e.g. $\pi = p$ or $\pi = t$, respectively), so $k = \mathcal{O}/\mathfrak{m}$ is a finite field of order q. Let $G = \mathrm{PGL}_2(F)$ and $K = \mathrm{PGL}_2(\mathcal{O})$, a maximal compact subgroup of G. The quotient space G/K is a discrete set which can be identified as the set of vertices of the regular tree of degree $q + 1$ in the following way:

Let $V = F^2$ be the two dimensional vector space over F. An $\mathscr{O}$-submodule L of V is called an $\mathscr{O}$-lattice if it is finitely generated as an $\mathscr{O}$-module and spans V over F. Every such L is of the form $L = \mathscr{O}\alpha + \mathscr{O}\beta$ where $\{\alpha, \beta\}$ is some basis of V over F. The *standard lattice* is the one with $\{\alpha, \beta\} = \{e_1, e_2\}$, where $\{e_1, e_2\}$ is the standard basis of V.

Two $\mathscr{O}$-lattices L_1 and L_2 are said to be equivalent if there exists $0 \neq \lambda \in F$ such that $L_2 = \lambda L_1$. The group $\mathrm{GL}_2(F)$ acts transitively on the set of $\mathscr{O}$-lattices and its center Z, the group of scalar matrices, preserves the equivalent classes. Hence $G = \mathrm{PGL}_2(F)$ acts on these classes, with $K = \mathrm{PGL}_2(\mathscr{O})$ fixing the equivalent class of the standard lattice $x_0 = [L_0]$, $L_0 = \mathscr{O}e_1 + \mathscr{O}e_2$. So, G/K can be identified with the set of equivalent classes of lattices. Two classes $[L_1]$ and $[L_2]$ are said to be *adjacent* if there exist representatives $L_1' \in [L_1]$ and $L_2' \in [L_2]$ such that $L_1' \subseteq L_2'$ and $L_2'/L_1' \simeq k(= \mathscr{O}/\mathfrak{m})$. This symmetric relation (since $\pi L_2' \subseteq L_1'$ and $L_1'/\pi L_2' \simeq k$) defines a structure of a graph.

Theorem 1.2.1 (*cf.* [Ser80, p. 70]). *The above graph is a* $(q+1)$*-regular tree.*

The $q+1$ neighbors of $[L_0]$ correspond to the $q+1$ subspaces of co-dimension 1 of the two dimensional space $L_0/\pi L_0 \cong k^2$. We can therefore identify them with $\mathbb{P}^1(k)$, the projective line over k.

Let us now shift our attention for a moment to the unitary representation theory of G. Let $C = C_c(K\backslash G/K)$ denote the set of bi-K-invariant functions on G with compact support. This is an algebra with respect to convolution:

$$f_1 * f_2(x) = \int_G f_1(xg) f_2(g^{-1})\, dg.$$

The algebra C is commutative (see [Lub94, Chapter 5] and the references there-in). If $\mathscr{H}$ is a Hilbert space and $\rho : G \to U(\mathscr{H})$ a unitary representation of G, then ρ induces a representation $\overline{\rho}$ of the algebra C by:

$$\overline{\rho}(f) = \int_G f(g)\rho(g)\, dg.$$

Let $\mathscr{H}^K$ be the space of K-invariant vectors in $\mathscr{H}$. Then $\overline{\rho}(f)(\mathscr{H}^K) \subseteq \mathscr{H}^K$ and so $(\mathscr{H}^K, \overline{\rho})$ is a representation of C. A basic claim is that if ρ is irreducible and $\mathscr{H}^K \neq \{0\}$ then $(\mathscr{H}^K, \overline{\rho})$ is irreducible. In fact, as C is commutative, Schur's Lemma implies that $\dim \mathscr{H}^K = 1$. So $\dim \mathscr{H}^K = 0$ or 1, in the second case we say that ρ is *K-spherical* (or unramified or of class one). We will be interested only in these representations. Such a representation ρ is uniquely determined by $(\mathscr{H}^K, \overline{\rho})$. Let us understand now what is the algebra C.

Let $\overline{\delta}$ be the characteristic function of the subset $K \left(\begin{smallmatrix} \pi & 0 \\ 0 & 1 \end{smallmatrix}\right) K$ of G. By its definition $\overline{\delta} \in C$. In fact, it turns out that C is generated as an algebra by $\overline{\delta}$ and hence every K-spherical irreducible subrepresentation $(\mathscr{H}, \rho)$ of G is determined by the action of $\overline{\delta}$ on the one dimensional space $\mathscr{H}^K$, i.e., by the eigenvalue of this action.

Let us note now that C also acts on $L^2(G/K)$ in the following way: If $f_1 \in C$ and $f_2 \in L^2(G/K)$ we think of both as functions on G and we can then look at $f_2 * f_1 \in L^2(G/K)$ (check!)

Spelling out the meaning of that for $f_1 = \overline{\delta}$, one can see (the reader is strongly encouraged to work out this exercise!):

Claim 1.2.2. Let f be a function defined on the vertices of the tree G/K and let δ be the operator $\delta : L^2(G/K) \to L^2(G/K)$ defined by $\delta(f) = f * \overline{\delta}$. Then for every $x \in G/K$

$$\delta(f)(x) = \sum_{y \sim x} f(y).$$

Namely, δ is nothing more then the adjacency operator (whose name in the classical literature is Hecke operator).

Let now Γ be a cocompact discrete subgroup of $G = \mathrm{PGL}_2(F)$ (for simplicity assume also that Γ is torsion free). Then $\Gamma \backslash G/K$ is, on one hand a quotient of the $(q+1)$-regular tree and, on the other hand, a quotient of the compact space $\Gamma \backslash G$. Hence, this is nothing more than a finite $(q+1)$-regular graph. Moreover, the discussion above shows that the spectral decomposition of the adjacency matrix of this finite graph (and in particular its eigenvalues) is intimately connected with the spectral decomposition of $L^2(G/K)$ as a unitary G-representation. More precisely, in every irreducible K-spherical subrepresentation ρ of $L^2(\Gamma \backslash G)$, there is a K-invariant function f, i.e., a function in $L^2(\Gamma \backslash G/K)$. As explained above, the one dimensional space spanned by f is a representation space $\overline{\rho}$ for C, which means that f is an eigenvector for the adjacency operator δ of the finite graph $\Gamma \backslash G/K$. Moreover, every eigenvector f of δ in $L^2(\Gamma \backslash G/K)$ is obtained like that (we can look at the G-subspace spanned by f, thinking of it as a K-invariant function in $L^2(\Gamma \backslash G)$.)

The list of K-spherical irreducible unitary representations of $\mathrm{PGL}_2(F)$ is well known (see [Lub94, Theorem 5.4.3] and the references therein). There are representations of two kinds:

(a) The tempered representations—these are the K-spherical irreducible representations $(\mathscr{H}, \rho)$ with the following property: There exist $0 \neq u, v \in \mathscr{H}$ such that $\phi : G \to \mathbb{C}$ defined by $\phi(g) = \langle \rho(g)u, v \rangle$ (the coefficient function of ρ with respect to u and v) is in $L^{2+\varepsilon}(G)$ for every $\varepsilon > 0$. The K-spherical representations with this property are also called in this case "the principal series" and they are characterized by the property that the associated eigenvalue λ of δ (as a generator of C acting on the one dimensional space $\mathscr{H}^K$) satisfies $|\lambda| \leq 2\sqrt{q}$.

(b) The non-tempered representations—these are the representations for which the above λ satisfies $2\sqrt{q} < |\lambda| \leq q+1$.

The above description explains why and how the representation of $G = \mathrm{PGL}_2(F)$ on $L^2(\Gamma\backslash G)$ is crucial for understanding the combinatorics of the graph $\Gamma\backslash G/K$. In fact we have (see [Lub94, Corollary 5.5.3]):

Theorem 1.2.3. *Let Γ be a cocompact lattice in $G = \mathrm{PGL}_2(F)$. Then $\Gamma\backslash G/K$ is a Ramanujan graph if and only if every irreducible K-spherical G-subrepresentation of $L^2(\Gamma\backslash G)$ is tempered, with the exception of the trivial representation (which corresponds to $\lambda = q + 1$) and the possible exception of the sign representation (the non-trivial one dimensional representation* $\mathrm{sg} : G \to \{\pm 1\}$*) which corresponds to $\lambda = -(q+1)$ and which appears in $L^2(\Gamma\backslash G)$ if and only if $\Gamma\backslash G/K$ is bipartite.*

Proving that Γ's as in the last theorem indeed exist is a highly non-trivial issue which we discuss in the next section. This will lead to (explicit) constructions of Ramanujan graphs.

Remark 1.2.4. In case Γ is a non-uniform lattice in $G = \mathrm{PGL}_2(F)$ (which exists only if $\mathrm{char}(F) > 0$) one can develop also a theory of Ramanujan diagrams (*cf.* [Mor94b]) which is also of interest even for computer science (see [Mor95]).

1.3. Explicit constructions

In this section we will quote the deep results which imply that various graphs are Ramanujan and then we will show how to use them to get explicit constructions of such graphs.

Let k be a global field, i.e., k is a finite extension of $\mathbb{Q}$ or of $\mathbb{F}_p(t)$. Let $\mathcal{O}$ be the ring of integers of k, S a finite set of valuations of k (containing all the Archimedean ones if $\mathrm{char}(k) = 0$) and $\mathcal{O}_S$ the ring of S-integers ($= \{x \in k \mid \nu(x) \geq 0, \forall \nu \notin S\}$). Let $\underline{G}$ be a k-algebraic semisimple group with a fixed embedding $\underline{G} \hookrightarrow \mathrm{GL}_m$. A general result asserts that

$$\Gamma = \underline{G}(\mathcal{O}_S) := \underline{G}(k) \cap \mathrm{GL}_m(\mathcal{O}_S)$$

is a lattice ($=$ discrete subgroup of finite covolume) in $\prod_{\nu \in S} \underline{G}(k_\nu)$ where k_ν is the completion of k with respect to the valuation ν. In some cases (few of these will be described below) $\underline{G}(k_\nu)$ is compact for every $\nu \in S$ except of one $\nu_0 \in S$. In such a case the projection of Γ to $\underline{G}(k_{\nu_0})$, which is also denoted by Γ, is called *an arithmetic lattice in* $\underline{G}(k_{\nu_0})$. The arithmetic lattice Γ comes with a system of congruence subgroups defined for every $0 \neq I \lhd \mathcal{O}_S$ as:

$$\Gamma(I) = \Gamma \cap \ker(\mathrm{GL}_m(\mathcal{O}_S) \longrightarrow \mathrm{GL}_m(\mathcal{O}_S/I)).$$

If $\underline{G}(k_{\nu_0}) \simeq \mathrm{PGL}_2(F)$ (or more generally $\mathrm{PGL}_d(F)$—see Chapter 2) where F is a local field as in §1.2, we get the arithmetic groups we are interested in. We can now state:

Theorem 1.3.1. *Let $\Gamma(I) \lhd \Gamma$ be a congruence subgroup of an arithmetic lattice Γ of $G = \mathrm{PGL}_2(F)$ as above. Then every infinite dimensional K-spherical irreducible subrepresentation of $L^2(\Gamma(I)\backslash G)$ is tempered.*

The only possible finite dimensional K-spherical representations are the trivial one and the sg representation. From Theorem 1.2.3 we can now deduce:

Corollary 1.3.2. *The graph $\Gamma(I)\backslash G/K$ is Ramanujan.*

Theorem 1.3.1 is a very deep result whose proof is a corollary of various works by some of the greatest mathematicians of the 20th century. It is based in particular on the solution of the so called "Peterson–Ramanujan conjecture". (In characteristic 0, in two steps: by Eichler for weight two modular forms which is the relevant case for us, and by Deligne in general. In positive characteristic by Drinfel'd.) Then one needs to combine it with the Jacquet–Langlands correspondence. The reader is referred to [Lub94] for more and in particular to the Appendix there by J. Rogawski which gives the general picture.

The last result gives explicit graphs in the mathematical sense of explicit, but it also paves the way for an explicit construction, in the computer science sense, of Ramanujan graphs. We will present the ones constructed in [LPS88].

When $G = \mathrm{PGL}_2(F)$, all the arithmetic lattices in G are obtained via quaternion algebras. Namely, let D be a quaternion algebra defined over k and $\underline{G} = D^\times/Z$, i.e., the invertible quaternions modulo the central ones. If D splits over $\nu_0 \in S$ (i.e., $D \otimes k_{\nu_0} \simeq M_2(k_{\nu_0})$) while it is ramified over all $\nu \in S\backslash\{\nu_0\}$ (i.e., $D \otimes k_\nu$ is a division algebra in which case $(D \otimes k_\nu)^\times/Z(D \otimes k_\nu)$ is a compact group) then $\underline{G}(\mathcal{O}_S)$ gives rise to an arithmetic lattice in $\underline{G}(k_{\nu_0}) = \mathrm{PGL}_2(k_{\nu_0})$. Such lattices (and their congruence subgroups) can be used for the construction of Ramanujan graphs.

Let us take a very concrete example: Let D be the classical Hamilton quaternion algebra; so D is spanned over $\mathbb{Q}$ by $1, i, j$ and k with $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$. It is well known that it is ramified over $\mathbb{R}$ and over $\mathbb{Q}_2$ while splits over $\mathbb{Q}_p$ for every odd prime p, and that $\underline{G}(\mathbb{R}) = \mathbb{H}^\times/\mathbb{R}^\times \simeq \mathrm{SO}(3)$ while $\underline{G}(\mathbb{Q}_p) = M_2(\mathbb{Q}_p)^\times/\mathbb{Q}_p^\times \simeq \mathrm{PGL}_2(\mathbb{Q}_p)$. Fix such a prime p and set $S = \{\nu_p, \nu_\infty\}$. Then $\mathcal{O}_S = \mathbb{Z}[\frac{1}{p}] = \{\frac{a}{p^n} \mid a \in \mathbb{Z}, n \in \mathbb{N}\}$, and as explained above, $D(\mathbb{Z}[\frac{1}{p}])$ is a discrete subring of $D(\mathbb{R}) \times D(\mathbb{Q}_p)$, while

$$\Gamma = D\Big(\mathbb{Z}\Big[\frac{1}{p}\Big]\Big)^\times/Z \hookrightarrow \mathrm{SO}(3) \times \mathrm{PGL}_2(\mathbb{Q}_p)$$

is a cocompact lattice.

Moreover, Jacobi's classical theorem tells us that there are $8(p+1)$ solutions to the equation: $x_0^2 + x_1^2 + x_2^2 + x_3^2 = p$ with $(x_0, x_1, x_2, x_3) \in \mathbb{Z}^4$. Assume now $p \equiv 1 (\mathrm{mod}\ 4)$. In this case three of the x_i are even and one is odd. If we agree to take those with x_0 odd and positive, we have a set Σ of $p+1$ solutions which come in pairs $\alpha_1, \overline{\alpha_1}, \ldots, \alpha_s, \overline{\alpha_s}$ where $s = \frac{p+1}{2}$ and where we consider α as an integral quaternion $\alpha = x_0 + x_1 i + x_2 j + x_3 k$, $\overline{\alpha} = x_0 - x_1 i - x_2 j - x_3 k$ and so $\|\alpha_i\| = \|\overline{\alpha_i}\| = p$. These $p+1$ elements give $p+1$ elements in $\underline{G}(\mathbb{Q}_p)$. Moreover, each α_i (and $\overline{\alpha_i}$) when considered as an element of $\mathrm{PGL}_2(\mathbb{Q}_p)$ takes the standard $\mathbb{Z}_p$-lattice in $\mathbb{Q}_p \times \mathbb{Q}_p$ (see §1.2) to an immediate neighbor in the tree and $\overline{\alpha_i} = \alpha_i^{-1}$. One can deduce (see [Lub94, Corollary 2.1.11]) that the group $\Lambda = \langle \alpha_1, \overline{\alpha_1}, \ldots, \alpha_s, \overline{\alpha_s} \rangle$ is a free group on $s = \frac{p+1}{2}$ generators acting simply transitive on the Bruhat–Tits $(p+1)$-regular tree T. One can therefore identify this tree with the Cayley graph of Λ with respect to $\Sigma = \{\alpha_i \mid i = 1, \ldots, s\}$. As Λ is also a lattice in $\mathrm{PGL}_2(\mathbb{Q}_p)$, it is of finite index in Γ. One can check (using "strong approximation" or directly) that if q is another prime with $q \equiv 1 (\mathrm{mod}\ 4)$ then $\Gamma(2q) \backslash T = \mathrm{Cay}(\Lambda/(\Lambda \cap \Gamma(2q)); \Sigma)$.

Spelling out the meaning of this, one gets the following explicit construction of Ramanujan graphs, which are usually referred to as the LPS-graphs.

Theorem 1.3.3 ([LPS88], see [Lub94, Theorem 7.4.3]). *Let p and q be different prime numbers with $p \equiv q \equiv 1 (\mathrm{mod}\ 4)$. Fix $\varepsilon \in \mathbb{F}_q$ satisfying $\varepsilon^2 = -1$. For each $\alpha_i = (x_0, x_1, x_2, x_3)$, $i = 1, \ldots, s$ in the set Σ above, take the matrix*

$$\widetilde{\alpha_i} = \begin{pmatrix} x_0 + \varepsilon x_1 & x_2 + \varepsilon x_3 \\ -x_2 + \varepsilon x_3 & x_0 - \varepsilon x_1 \end{pmatrix} \in \mathrm{PGL}_2(\mathbb{F}_q)$$

and $\widetilde{\Sigma} = \{\widetilde{\alpha_i} \mid i = 1, \ldots, s\}$. Let H be the subgroup of $\mathrm{PGL}_2(\mathbb{F}_q)$ generated by $\widetilde{\Sigma}$ and $X^{p,q} = \mathrm{Cay}(H; \widetilde{\Sigma})$, the Cayley graph of H with respect to $\widetilde{\Sigma}$. Then:

(a) $X^{p,q}$ is a $(p+1)$-regular Ramanujan graph.
(b) If $(\frac{p}{q}) = -1$, i.e., p is not a quadratic residue modulo q, then $H = \mathrm{PGL}_2(\mathbb{F}_q)$ and $X^{p,q}$ is a bi-partite graph, while if $(\frac{p}{q}) = 1$, $H = \mathrm{PSL}_2(\mathbb{F}_q)$ and $X^{p,q}$ is not bi-partite.

The main motivation for the construction of Ramanujan graphs has been expanders, but the LPS graphs turned out to have various other remarkable properties like high girth and high chromatic numbers (simultaneously!). See [LPS88, Lub94, Sar90, Val97] for more.

In [Mor94a], Morgenstern constructed, for every prime power q, infinitely many $(q+1)$-regular Ramanujan graphs. This time by finding an arithmetic lattice in $\mathrm{PGL}_2(\mathbb{F}_q((t)))$ acting simply transitive on the Bruhat–Tits tree. Another such a construction is given (somewhat hidden) in [LSV05a] as a special case of Ramanujan complexes (to be discussed in the next chapter). These last mentioned Ramanujan graphs are also edge transitive and not merely vertex transitive (see [KL12]).

2. Ramanujan complexes

This chapter is devoted to the high-dimensional version of Ramanujan graphs, the so called Ramanujan complexes. These are $(d-1)$-dimensional simplicial complexes which are obtained as quotients of the Bruhat–Tits building $\mathscr{B}_d$ associated with $\mathrm{PGL}_d(F)$, F a local field, just like the Ramanujan graphs were obtained as quotients of the Bruhat–Tits tree of $\mathrm{PGL}_2(F)$. What enables this, is the work of Lafforgue [Laf02] which extended to general d, the "Ramanujan conjecture" for GL_d in positive characteristic, proved first by Drinfel'd [Dri88] for $d = 2$ (the work of Drinfel'd was the basis for the Ramanujan graphs constructed by Morgenstern [Mor94a]). We will start in §2.1 with the basic definitions and results about buildings and will present the associated Hecke operators, generalizing the Hecke operator (=adjacency matrix) which appeared in Chap. 1. We will present the analogue of Alon–Boppana Theorem and define Ramanujan complexes. In §2.2 we will survey shortly the representation theory of $\mathrm{PGL}_d(F)$ and just as in Theorem 1.2.3, we will show that representation theory is relevant for the combinatorics of $\Gamma\backslash\mathscr{B}_d$. Then in §2.3, we will present explicit constructions of Ramanujan complexes.

We will follow mainly [LSV05a] and [LSV05b], where the reader can find precise references for the results mentioned here. The reader is also referred to [Bal00, CSŻ03, Li04, Sar07] for related material.

2.1. Bruhat–Tits buildings and Hecke operators

Let K be any field. The spherical complex $\mathbb{P}^{d-1}(K)$ associated with K^d is the simplicial complex whose vertices are all the non-trivial (i.e., not $\{0\}$ and not K^d) subspaces of K^d. Two subspaces W_1 and W_2 are on the same 1-edge if either $W_1 \subseteq W_2$ or $W_2 \subseteq W_1$, and $\{W_0, \dots, W_r\}$ is an r-cell if every pair W_i, W_j forms an edge (i.e., $\mathbb{P}^{d-1}(K)$ is a "clique complex"). It can be shown that this happens if and only if after some reordering $W_0 \subseteq W_1 \subseteq \cdots \subseteq W_r$. When $K = \mathbb{F}_q$, a finite field of order q, $\mathbb{P}^1(\mathbb{F}_q)$ is just a set of $q+1$ points (which can be identified with the projective line over $\mathbb{F}_q$). For $d = 3$, $\mathbb{P}^2(K)$ is the $(q+1)$-regular graph with $2(q^2+q+1)$ vertices of "points" versus "lines" of the projective plane. In general $\mathbb{P}^{d-1}(K)$ is a $(d-2)$-dimensional simplicial complex.

We now describe $\mathscr{B} = \mathscr{B}_d(F)$, the affine Bruhat–Tits building of type $\widetilde{A}_{d-1}$ associated with F. Here F is a local field with $\mathscr{O}$, π and $\mathfrak{m}$ as in §1.2 and $\mathscr{O}/\mathfrak{m} = \mathbb{F}_q$. An $\mathscr{O}$-lattice in $V = F^d$ is a finitely generated $\mathscr{O}$-submodule L of V such that L contains an F-basis of V. Two lattices L_1 and L_2 are equivalent if $L_1 = \lambda L_2$ for some $0 \neq \lambda \in F$. The vertices of $\mathscr{B}$ are the equivalence classes of $\mathscr{O}$-lattices in V, and two distinct equivalent classes $[L_1]$ and $[L_2]$ are adjacent in $\mathscr{B}$ if there exist representatives $L_1' \in [L_1]$ and $L_2' \in [L_2]$ s.t. $\pi L_1' \subseteq L_2' \subseteq$

L'_1. The r-simplices of $\mathscr{B}$ ($r \geq 2$) are the subsets $\{[L_0], \dots, [L_r]\}$ such that all pairs $[L_i]$ and $[L_j]$ are adjacent. It can be shown that $\{[L_0], \dots, [L_r]\}$ forms a simplex if and only if there exist representatives $L'_i \in [L_i]$ such that after reordering the indices, $\pi L'_r \subseteq L'_0 \subseteq \cdots \subseteq L'_r$. The complex $\mathscr{B}$ is therefore of dimension $d-1 = \operatorname{rank}_F(\mathrm{PGL}_d(F))$. This is a special case of the Bruhat–Tits building associated with a reductive group over F. The next theorem is also a special case which generalizes Theorem 1.2.1:

Theorem 2.1.1. *The complex $\mathscr{B}_d(F)$ is contractible. The link of each vertex is isomorphic to $\mathbb{P}^{d-1}(\mathbb{F}_q)$.*

The vertices of $\mathscr{B}$ come with a natural coloring ("type"). Let $\tau : \mathscr{B}^0 \to \mathbb{Z}/d\mathbb{Z}$ be defined as follows: Let $\mathscr{O}^d \subseteq V$ be the standard lattice in V. For any lattice L, there exists $g \in GL(V) = GL_d(F)$ such that $L = g(\mathscr{O}^d)$. Define $\tau([L]) = \nu(\det(g))(\bmod d)$ where ν is the valuation of F, e.g. for $F = \mathbb{F}_q((t))$, $\nu(\sum_{i=m}^{\infty} a_i t^i) = m$ when $m \in \mathbb{Z}$ and $a_m \neq 0$. Any two vertices in any simplex of $\mathscr{B}$ have different colors.

The group $G = \mathrm{GL}_d(F)$ acts transitively on the $\mathscr{O}$-lattices in V and its center preserves the equivalence classes. As the action preserves inclusions, $G = \mathrm{PGL}_d(F)$ acts on the building $\mathscr{B}$. It acts transitively on $\mathscr{B}^0$—the vertices—without preserving their colors, but $\mathrm{PSL}_d(F)$ does preserves them. The stabilizer of the (equivalence class of the) standard lattice is $K = \mathrm{PGL}_d(\mathscr{O})$. Hence $\mathscr{B}^0$ can be identified with G/K. To every directed edge $(x, y) \in \mathscr{B}^1$ one can associate the color $\tau(y) - \tau(x) \in \mathbb{Z}/d\mathbb{Z}$. The color of edges is preserved by $\mathrm{PGL}_d(F)$.

Let us now define $d-1$ operators—the Hecke operators—as follows: For $1 \leq k \leq d-1$, $f \in L^2(\mathscr{B}^0)$ and $x \in \mathscr{B}^0$,

$$(A_k f)(x) = \sum f(y), \tag{2.1}$$

where the summation is over the neighbors y of x such that $\tau(y) - \tau(x) = k \in \mathbb{Z}/d\mathbb{Z}$. If $x = [L]$, L a lattice, then this amounts to a sum over the sublattices of L containing πL, whose projection in $L/\pi L$ are of codimension k there. Note that A_k commutes with the action of $\mathrm{PGL}_d(F)$. One can show that these operators are bounded, normal and commute with each other. But in general they are not self-adjoint. In fact, $A_k^* = A_{d-k}$ so $A_k + A_{d-k}$ is self-adjoint. Moreover $\Delta = \sum_{k=1}^{d-1} A_k$ is the adjacency operator of the 1-skeleton of $\mathscr{B}$. For $d = 2$, we only have $A_1 = A_1^*$ which is indeed the adjacency operator of the tree. As the operators A_k commute with each other they can be diagonalized simultaneously. Their common spectrum is therefore a subset Σ_d of $\mathbb{C}^{d-1}$.

Theorem 2.1.2. *Let $S = \{(z_1, \dots, z_d) \in \mathbb{C}^d \mid |z_i| = 1 and \prod_{i=1}^d z_i = 1\}$ and $\sigma : (z_1, \dots, z_d) \mapsto (\lambda_1, \dots, \lambda_{d-1})$ where $\lambda_k = q^{\frac{k(d-k)}{2}} \sigma_k(z_1, \dots, z_d)$. Here*

σ_k is the k-th elementary symmetric function, i.e., $\sigma_k(z_1,\dots,z_d) = \sum_{i_i<\dots<i_k} z_{i_1}\cdot\dots\cdot z_{i_k}$. Then $\sigma(S)$ is equal to Σ_d, the simultaneous spectrum of $A_1,\dots,A_{d-1}$ acting on $L^2(\mathscr{B}^0)$.

Note that indeed $\lambda_k = \overline{\lambda_{d-k}}$ as had to be expected, since $A_k = A^*_{d-k}$. Also for $d = 2$,

$$\Sigma_2 = \sigma(S) = \left\{q^{\frac{1}{2}}\left(z+\frac{1}{z}\right)\,\middle|\, z\in\mathbb{C}, |z|=1\right\} = [-2\sqrt{q}, 2\sqrt{q}]$$

which shows that Theorem 2.1.2 is a generalization of Proposition 1.1.3.

Ramanujan $(q+1)$-regular graphs were defined as the finite quotients of $\mathscr{B}_2 = T_{q+1}$ whose "non-trivial" eigenvalues are all in Σ_2. Similarly we will define Ramanujan complexes as quotients of $\mathscr{B}_d$ whose "non-trivial" eigenvalues are in Σ_d. Let us describe first the trivial eigenvalues: Recall that for $d = 2$ we have two such: $(q+1)$ and $-(q+1)$. They appear in all the finite quotients $\Gamma\backslash\mathscr{B}_2$ when Γ preserves the colors of the vertices (and only $q+1$ appears in all the finite quotients).

So, assume $\Gamma \le \mathrm{PGL}_d(F)$ is a cocompact lattice preserving the colors of the vertices of $\mathscr{B}^0$. So, τ is well defined on $X = \Gamma\backslash\mathscr{B}^0$. For a d-th root of unity ξ, define $f_\xi : X \to \mathbb{C}$ by $f_\xi(x) = \xi^{\tau(x)}$. Now, $A_k f_\xi(x)$ sums over the neighbors of x of color $\tau(x)+k(\operatorname{mod} d)$ and there are $\left[\begin{smallmatrix} d\\ k\end{smallmatrix}\right]_q$ vertices like that (where $\left[\begin{smallmatrix} d\\ k\end{smallmatrix}\right]_q$ denotes the number of subspaces of codimension k in $\mathbb{F}_q^d$). Hence $A_k f_\xi(x) = \left[\begin{smallmatrix} d\\ k\end{smallmatrix}\right]_q \xi^{\tau(x)+k} = \left[\begin{smallmatrix} d\\ k\end{smallmatrix}\right]_q \xi^k f_\xi(x)$. Thus, for every $\xi\in\mathbb{C}$ with $\xi^d = 1$, we get a simultaneous "trivial" eigenvalue $(\left[\begin{smallmatrix} d\\ 1\end{smallmatrix}\right]_q \xi^1,\dots,\left[\begin{smallmatrix} d\\ k\end{smallmatrix}\right]_q \xi^k,\dots,\left[\begin{smallmatrix} d\\ d-1\end{smallmatrix}\right]_q \xi^{d-1})$. These are the d trivial eigenvalues. Again, for $d = 2$, we get $\left[\begin{smallmatrix} 2\\ 1\end{smallmatrix}\right]_q\cdot 1 = q+1$ and $\left[\begin{smallmatrix} 2\\ 1\end{smallmatrix}\right]_q(-1) = -(q+1)$. We can now define:

Definition 2.1.3. *A Ramanujan complex (of type $\widetilde{A}_{d-1}$) is a finite quotient $X = \Gamma\backslash\mathscr{B}_d$ satisfying: every simultaneous eigenvalue $(\lambda_1,\dots,\lambda_k,\dots,\lambda_{d-1})$ of $(A_1,\dots,A_k,\dots,A_{d-1})$ satisfies: either $(\lambda_1,\dots,\lambda_{d-1})$ is one of the d trivial eigenvalues (i.e., $(\lambda_1,\dots,\lambda_{d-1}) = (\left[\begin{smallmatrix} d\\ 1\end{smallmatrix}\right]_q \xi^1,\dots,\left[\begin{smallmatrix} d\\ d-1\end{smallmatrix}\right]_q \xi^{d-1})$ for some d-th root of unity ξ) or $(\lambda_1,\dots,\lambda_{d-1}) \in \Sigma_d$, described in Theorem 2.1.2.*

In the case of $d = 2$, we saw the Alon–Boppana Theorem (Theorem 1.1.2) which shows that the Ramanujan bounds are the strongest one can hope from an infinite family of $(q+1)$-regular graphs (for a fixed q). The following theorem is a strong high dimensional version.

Theorem 2.1.4 ([Li04, Theorem 4.3]). *Let X_i be a family of finite quotients of $\mathscr{B}_d$ with unbounded injective radius (recall that the injective radius of a quotient $\pi : \mathscr{B}\to\Gamma\backslash\mathscr{B}$ is the maximal r such that π is an isomorphism when restricted to any ball of radius r in $\mathscr{B}$). Then $\overline{\cup\mathrm{spec}_{X_i}(A_1,\dots,A_{d-1})} \supseteq \Sigma_d$.*

Note that $\operatorname{spec}_{X_i}(A_1, \dots, A_{d-1})$ is a finite set for every i. So the best bounds we can hope for X_i's, as a family, is to be Ramanujan. The reader is referred to [Fi14] for a very general "Alon–Boppana theorem". This work also gives a meaning to "Ramanujan at level i" for every $0 \le i < d-1$, while we consider here only level 0.

Let us end this section with the following remark:

Remark 2.1.5. The trivial eigenvalues of $(A_1, \dots, A_{d-1})$ are $(\lambda_1, \dots, \lambda_{d-1}) = (\left[{d \atop 1}\right]_q \xi^1, \dots, \left[{d \atop d-1}\right]_q \xi^{d-1})$. So for A_k, $|\lambda_k| = \left[{d \atop k}\right]_q \approx q^{k(d-k)}$ while the Ramanujan bound gives:

$$|\lambda_k| \le q^{\frac{k(d-k)}{2}} |\sigma_k(z_1, \dots, z_k)| \le \binom{d}{k} q^{\frac{k(d-k)}{2}}$$

so for d fixed and q large, the Ramanujan bound is approximately the square root of the trivial eigenvalue.

In §1.1 we mentioned that Ramanujan graphs can be characterized by the fact that their zeta functions satisfy the Riemann hypothesis. Recently there have been some efforts to associate zeta functions to higher dimensional complexes with the hope to give a similar characterization for Ramanujan complexes of dimension 2. See [DH06, Sto06, KLW10, DK14, KL14]. It will be nice if this theory could be extended also to higher dimensions.

2.2. *Representation theory of* PGL_d

In this section we will describe some basic results from the representation theory of $\mathrm{PGL}_d(F)$, F a local field. For a more comprehensive survey see [Car79]. We will give only those results which are needed for our combinatorial application. The goal is to get a high dimensional generalization of Theorem 1.2.3, i.e., a representation theoretic formulation of Ramanujan complexes.

Let $G = \mathrm{PGL}_d(F)$ and $K = \mathrm{PGL}_d(\mathcal{O})$, $\mathcal{O}$ the ring of integers of F. An irreducible unitary representation $(\mathcal{H}, \rho)$ of G is called K-spherical if the space of K-fixed points $\mathcal{H}^K$ is non-zero. In this case $\dim \mathcal{H}^K = 1$. Let $C = C_c(K \backslash G / K)$ be the algebra of compactly supported bi-K-invariant functions from G to $\mathbb{C}$, with multiplication defined by convolution

$$f_1 * f_2(x) = \int_G f_1(xg) f_2(g^{-1})\, dg.$$

The algebra C is called the Hecke algebra of G. Let $\widetilde{\pi}_k = \operatorname{diag}(\pi, \pi, \dots, \pi, 1, 1, \dots, 1) \in \mathrm{GL}_d(F)$ with $\det(\widetilde{\pi}_k) = \pi^{d-k}$, where π is the uniformizer of F. Denote by π_k the image of $\widetilde{\pi}_k$ in $\mathrm{PGL}_d(F)$ and let A_k be the characteristic function of $K \pi_k K$. Clearly $\{A_k\}_{k=1}^{d-1} \subseteq C$ (note $\pi_0 = \pi_d = I_d$). Less trivial is the fact that C is commutative and is freely generated as a commutative algebra by $A_1, \dots, A_{d-1}$ (*cf.* [Mac79, Chap. V]). Every irreducible unitary representation $(\mathscr{H}, \rho)$ of G gives rise to a representation of C on $\mathscr{H}^K$ and when $\mathscr{H}^K \neq \{0\}$, this last representation is in fact given by a homomorphism $w : C \to \mathbb{C}$, $f \cdot v_0 = w(f)v_0$ for $f \in C$. The representation ρ is uniquely determined by w (*cf.* [LSV05a, Prop. 2.2]) and w is determined by the $(d-1)$-tuple $(w(A_1), \dots, w(A_{d-1})) \in \mathbb{C}^{d-1}$.

Let us put this in a somewhat more known formulation: a more common parametrization of the irreducible spherical representations of $\mathrm{GL}_d(F)$ (and hence also of $\mathrm{PGL}_d(F)$) is by their *Satake parametrization* $(z_1, \dots, z_d) \in (\mathbb{C}^\times)^d / Sym(d)$. This parametrization is related but not the same as the one we discuss here. Let us just mention here that

(a) A representation of $\mathrm{GL}_d(F)$ with Satake parameters $(z_1, \dots, z_k)$ factors through $\mathrm{PGL}_d(F)$ if and only if $\prod_{i=1}^d z_i = 1$.
(b) If $(\mathscr{H}, \rho)$ is an irreducible spherical representation of $\mathrm{PGL}_d(F)$ with Satake parameters $(z_1, \dots, z_d)$ then $w(A_k)$ in the notation above is given by $w(A_k) = q^{\frac{k(d-k)}{2}} \sigma_k(z_1, \dots, z_d)$ where σ_k is the k-th elementary symmetric function on d variables, $\sigma_k(z_1, \dots, z_d) = \sum_{i_i < \dots < i_k} z_{i_1} \cdot \cdots \cdot z_{i_k}$.
(c) An irreducible representation $(\mathscr{H}, \rho)$ is called *tempered*, if there exists $0 \neq v, u \in \mathscr{H}$ such that the coefficient function $\psi(g) = \langle \rho(g)v, u \rangle$ is in $L^{2+\varepsilon}(G)$ for every $\varepsilon > 0$. These are exactly the representations which are *weakly contained* (in the sense of the Fell topology) in the representation of G on $L^2(G)$. If such a representation is also K-spherical then it is weakly contained in the representation of G on $L^2(G/K) = L^2(\mathscr{B}_0)$. In terms of Satake parameters, ρ is tempered if and only if $|z_i| = 1$ for all i.

The reader is referred to more information in [LSV05a] and for the general theory in [Car79]. At this point, especially in light of (b) and (c) the reader may start to guess the connection to Ramanujan complexes. Let us spell it out explicitly.

Let $L_0 = \mathscr{O}e_1 + \cdots + \mathscr{O}e_d$ be the standard $\mathscr{O}$-lattice in $V = F^d$ and $[L_0]$ its equivalence class, which corresponds to K under the identification $G/K = \mathscr{B}^0$. Let Ω_k be the set of neighbors of color k of $[L_0]$. Then $\pi_k^{-1} K \in G/K = \mathscr{B}^0$ is one of these neighbors and K (as a subgroup of G) acts transitively on Ω_k so that $K\pi_k^{-1}K = \cup yK$ where the union is over all $yK \in \Omega_k$. Multiplying from the left by an arbitrary $g \in G$, we see that the neighbors of the vertex gK forming an edge of color k with it, are exactly $\{gyK\}_{yK \in \Omega_k}$. It follows that the operator A_k defined in (2.1) in §2.1, can be expressed as follows: Identifying $L^2(\mathscr{B}^0) = L^2(G/K)$ with the right K-invariant functions in $L^2(G)$, and assuming that K has Haar measure one, for $f \in L^2(\mathscr{B}^0)$, and $gK \in \mathscr{B}^0$

$$\begin{aligned}(A_k f)(gK) &= \sum_{yK \in \Omega_k} f(gyK) = \sum_{yK \in \Omega_k} \int_{yK} f(gz)\,dz \\ &= \int_{K\pi_k^{-1}K} f(gz)\,dz = \int_G f(gz)\mathbf{1}_{K\pi_k K}(z^{-1})\,dz = (f * A_k)(gK),\end{aligned} \tag{2.2}$$

where A_k at the right hand side of equation (2.2) is the characteristic function of $K\pi_k K$, as defined in this section. No confusion should occur here as Eq. (2.2) shows that the Hecke operators of §2.1 and the Hecke operators of §2.2 are essentially the same thing! When $C = C_c(K\backslash G/K)$ acts on $L^2(G/K)$, A_k acts as the adjacency operators summing over all the neighbors with edges of color k.

We can now use this to deduce the main goal of this subsection (see [LSV05a, Prop. 1.5]).

Proposition 2.2.1. *Let* Γ *be a cocompact lattice of* $\mathrm{PGL}_d(F)$. *Then* $\Gamma\backslash\mathscr{B}$ *is a Ramanujan complex if and only if every irreducible spherical infinite dimensional* G*-subrepresentation of* $L^2(\Gamma\backslash\mathrm{PGL}_d(F))$ *is tempered.*

Sketch of proof. Assume every irreducible spherical infinite dimensional subrepresentation of $\mathscr{H} = L^2(\Gamma\backslash\mathrm{PGL}_d(F))$ is tempered. As Γ is cocompact, $\mathscr{H}$ is a direct sum of irreducible representations. Let $f \in L^2(\Gamma\backslash G/K)$ be a non-trivial simultaneous eigenfunction of the Hecke operators A_k with $A_k f = \lambda_k f$. As $\mathrm{PSL}_d(F)$ has no non-trivial finite dimensional representations, every finite dimensional representation of $\mathrm{PGL}_d(F)$ factors through $\mathrm{PGL}_d(F)/\mathrm{PSL}_d(F) \simeq F^\times/(F^\times)^d$. Since $F^\times \simeq \mathbb{Z} \times \mathscr{O}^\times$, we have $F^\times/\mathscr{O}^\times(F^\times)^d \simeq \mathbb{Z}/d\mathbb{Z}$ and since f is fixed by K, if f lies in a finite dimensional G-subspace, it correspond to one of the d trivial eigenvalue. If f spans an infinite dimensional G-space, then it is tempered, its Satake parameters $(z_1, \dots, z_d)$ satisfy $\prod z_i = 1$ and $|z_i| = 1$. The corresponding eigenvalues of A_k are, as explained in point (b) above, in Σ_d as defined in §2.1.

In the other direction: If $\mathscr{H}_1$ is an irreducible spherical infinite dimensional subrepresentation of $L^2(\Gamma\backslash G)$, then its unique (up to scalar) K-fixed vector f is a simultaneous eigenvector of all the A_k's where $A_k f = \lambda_k f$. By assumption $(\lambda_1, \dots, \lambda_{d-1}) \in \Sigma_d$, from which we deduce that the Satake parameters z_i all satisfy $|z_i| = 1$ and the representation is tempered. □

So, once again, as we saw for Ramanujan graphs, the problem of constructing Ramanujan complexes moves from combinatorics to representation theory. In the next subsection, we will describe how deep results in the area of automorphic forms lead to such combinatorial constructions.

Remark 2.2.2. In analogy to the Ramanujan diagrams built out of non-uniform lattices in $\mathrm{PGL}_2(\mathbb{F}_q((t)))$—see Remark 1.2.4—one can build also non-uniform Ramanujan complexes (see [Sa09]).

2.3. Explicit construction of Ramanujan complexes

We will start with a general result which gives a lot of Ramanujan complexes. We then continue to present a very explicit construction.

Let us first recall some notations and add a few more: Let k be a global field of characteristic $p > 0$ and D a division algebra of degree d over k. Denote by $\underset{\sim}{G}$ the k-algebraic group $D^\times/k^\times$, and fix an embedding of $\underset{\sim}{G}$ into GL_n for some n. Let T be the finite set of valuations of k for which D does not split. We assume that for every $\nu \in T$, $D_\nu = D \otimes_k k_\nu$ is a division algebra. Let ν_0 be a valuation of k which is not in T and $F = k_{\nu_0}$, so that $\underset{\sim}{G}(F) \simeq \mathrm{PGL}_d(F)$, and denote $S = T \cup \{\nu_0\}$. For $\mathscr{O}_S = \{x \in k \mid \nu(x) \geq 0 \ \forall \nu \notin S\}$ the ring of S-integers in k, $\underset{\sim}{G}(\mathscr{O}_S) := \underset{\sim}{G}(k) \cap \mathrm{GL}_n(\mathscr{O}_S)$ embeds diagonally as a discrete subgroup of $\prod_{\nu \in S} \underset{\sim}{G}(k_\nu)$. Projecting this subgroup into $\underset{\sim}{G}(k_{\nu_0}) = \underset{\sim}{G}(F) \simeq \mathrm{PGL}_d(F)$ gives an embedding of $\underset{\sim}{G}(\mathscr{O}_S)$ as a discrete subgroup in $\mathrm{PGL}_d(F)$, which we denote by Γ. In fact, by a general result on arithmetic subgroups, Γ is a cocompact lattice in $\mathrm{PGL}_d(F)$. Thus if $\mathscr{B} = \mathscr{B}_d(F)$ is the Bruhat–Tits building associated with $\mathrm{PGL}_d(F)$, then $\Gamma \backslash \mathscr{B}$ is a finite complex. The same is true when we mod $\mathscr{B}$ by any finite index subgroup of Γ. In particular, if $0 \neq I \lhd \mathscr{O}_S$ is an ideal, then the congruence subgroup $\Gamma(I) := \ker(\underset{\sim}{G}(\mathscr{O}_S) \to \underset{\sim}{G}(\mathscr{O}_S/I))$ is of finite index in Γ and $\Gamma(I)\backslash \mathscr{B}$ is a finite simplicial complex covered by $\mathscr{B}$.

Theorem 2.3.1. *For Γ and I as above, $\Gamma(I)\backslash\mathscr{B}$ is a Ramanujan complex.*

A word of warning: if d is not a prime then there are ideals in $\mathscr{O}_{\nu_0} = \{x \in k \mid \nu(x) \geq 0 \ \forall \nu \neq \nu_0\}$ (so they may disappear in $\mathscr{O}_S$!) which give non-Ramanujan complexes. We refer to [LSV05a] for this delicate point as well as for a proof of Theorem 2.3.1. We will not try to explain the proof, but rather give few hints about it. The Theorem is proved there by going from local to global. By Proposition 2.2.1 above, $\Gamma(I)\backslash\mathscr{B}$ is Ramanujan if and only if every infinite dimensional irreducible spherical subrepresentation ρ_0 of $L^2(\Gamma(I)\backslash \mathrm{PGL}_d(F))$ is tempered. One shows that such ρ_0 is a local factor at ν_0 of an automorphic adelic subrepresentation ρ' of $L^2(\underset{\sim}{G}(k)\backslash \underset{\sim}{G}(\mathbb{A}))$ where $\mathbb{A}$ is the ring of adeles of k. By using the Jacquet–Langlands correspondence, one can replace ρ' by a suitable subrepresentation ρ of $L^2(\mathrm{PGL}_d(k)\backslash \mathrm{PGL}_d(\mathbb{A}))$. Then one appeals to the work of Lafforgue [Laf02] (for which he got the Fields medal!) which is an extension to general d of the "Ramanujan conjecture" proved by Drinfel'd for $d = 2$. This last result says that for various adelic automorphic representations, the local factors are tempered. This can be applied to ρ to deduce that our ρ_0 is tempered and hence $\Gamma(I)\backslash\mathscr{B}$ is Ramanujan.

The description of the complexes we gave is pretty abstract but it can be made very explicit in some cases. To this end we will make use (following [LSV05b]) of a remarkable arithmetic lattice Γ constructed by Cartwright and Steger [CS98]. This lattice has the following amazing property: It acts simply transitively on the vertices of the building $\mathscr{B}_d$. Such lattices are rare; for example in characteristic zero such lattices exist only for finitely many d's (see [MSG12])). Let us describe its (somewhat technical) construction:

We start with the global field $k = \mathbb{F}_q(y)$, whose valuations are ν_g for every irreducible monic polynomial g in $\mathbb{F}_q[y]$, and the *minus degree* valuation $\nu_{\frac{1}{y}}(f/g) = \deg g - \deg f$. Let $\mathbb{F}_{q^d}$ be the field extension of $\mathbb{F}_q$ of degree d and ϕ a generator of the Galois group $\mathrm{Gal}(\mathbb{F}_{q^d}/\mathbb{F}_q) \simeq \mathbb{Z}/d\mathbb{Z}$. Fix a basis $\xi_0, \dots, \xi_{d-1}$ of $\mathbb{F}_{q^d}$ over $\mathbb{F}_q$ with $\xi_i = \phi^i(\xi_0)$. Let D be the k-algebra with basis $\{\xi_i z^j\}_{i,j=0}^{d-1}$ and relations $z\xi_i = \phi(\xi_i)z$ and $z^d = 1+y$. Then D is a division algebra which ramifies at $T = \{\nu_{1+y}, \nu_{\frac{1}{y}}\}$ and splits at all other completions of k (see [LSV05b, Prop. 3.1]). That is, $D_{\nu_{1+y}} = D\otimes_k k_{\nu_{1+y}} = D\otimes_k \mathbb{F}_q((1+y))$ and $D_{\nu_{\frac{1}{y}}} = D \otimes_k \mathbb{F}_q((\frac{1}{y}))$ are division algebras, while $D_\nu \simeq M_d(k_\nu)$ for $\nu \notin T$. In particular, $\underline{G}(k_\nu) \simeq \mathrm{PGL}_d(k_\nu)$ for $\nu \notin T$, where we recall that $\underline{G}$ denotes the k-algebraic group $D^\times/k^\times$.

For ν_0 we take the valuation ν_y, which is given explicitly by $\nu_y(a_m y^m + \cdots + a_n y^n) = m$ ($a_m \neq 0, m \leq n$). The completion of k at ν_0 is $F = k_{\nu_y} = \mathbb{F}_q((y))$, the field of Laurent polynomials over $\mathbb{F}_q$. The ring of integer of F is $\mathscr{O} = \mathbb{F}_q[[y]]$, and we recall that $\mathscr{B}_d^0 \simeq \mathrm{PGL}_d(F)/\mathrm{PGL}_d(\mathscr{O})$.

We now have $S = \{\nu_{1+y}, \nu_{\frac{1}{y}}, \nu_y\}$, and the ring of S-integers in k is $\mathscr{O}_S = \mathbb{F}_q[\frac{1}{1+y}, y, \frac{1}{y}]$. As explained above, embedding $\underline{G}(k)$ in some $\mathrm{GL}_n(k)$ gives rise to $\Gamma = \underline{G}(\mathscr{O}_S) = \underline{G}(k) \cap \mathrm{GL}_n(\mathscr{O}_S)$, which embeds as a cocompact arithmetic lattice in $\underline{G}(F) \simeq \mathrm{PGL}_d(F)$.

Until now we have followed the general construction described in the beginning of this section. In what follows we describe the Cartwright–Steger group, a subgroup of Γ which acts simply transitively on $\mathscr{B}_d^0$.

The definition of $\Gamma = \underline{G}(\mathscr{O}_S)$ involves a choice of an embedding of $\underline{G}(k)$ in $\mathrm{GL}_n(k)$. It turns out that this embedding can be chosen so that Γ is simply $D(\mathscr{O}_S)^\times/\mathscr{O}_S^\times$, where $D(\mathscr{O}_S)$ stands for the $\mathscr{O}_S$-algebra having the $\mathscr{O}_S$-basis $\{\xi_i z^j\}_{i,j=0}^{d-1}$, and again the relations $z\xi_i = \phi(\xi_i)z$ and $z^d = 1 + y$ (see [LSV05b, Prop. 3.3]). Note that as $z^d = 1 + y$ and $1 + y$ is invertible in $\mathscr{O}_S$, z is invertible in $D(\mathscr{O}_S)$. Let $b = 1 - z^{-1} \in D(\mathscr{O}_S)$, and note that b is also invertible, since it divides $1 - z^{-d} = \frac{y}{1+y}$ and $y \in \mathscr{O}_S^\times$. Also note that $\mathbb{F}_{q^d}$ is a subring of $D(\mathscr{O}_S)$ spanned by the ξ_i's. For every $u \in \mathbb{F}_{q^d}^\times$ denote $b_u = ubu^{-1} \in D(\mathcal{O}_S)^\times$. The element b_u depends only on the coset of u in $\mathbb{F}_{q^d}^\times/\mathbb{F}_q^\times$, since $\mathbb{F}_q \subseteq Z(D(\mathcal{O}_S))$. Denoting by $\overline{b_u}$ the image of b_u in $\Gamma = D(\mathcal{O}_S)^\times/\mathcal{O}_S^\times$, this gives us a set of $\frac{q^d-1}{q-1}$ elements $\Sigma_1 = \{\overline{b_u} \mid u \in \mathbb{F}_{q^d}^\times/\mathbb{F}_q^\times\}$ in Γ. Let $\Lambda = \langle \Sigma_1 \rangle$. This is the promised Cartwright–Steger group.

Theorem 2.3.2 ([CS98], *cf.* [LSV05b, Prop. 4.8]). *The group Λ acts simply transitively on the vertices of $\mathcal{B}_d(F)$.*

The set $\Sigma_1 = \{\overline{b_u} \mid u \in \mathbb{F}_{q^d}^\times/\mathbb{F}_q^\times\}$ takes the "initial vertex" x_0 of the building (i.e., the equivalence class of the standard lattice) to the set of its neighbors x with $\tau(x) = 1$ (i.e., the neighboring vertices of color 1, for which the connecting edge also has color 1). These correspond to the codimension one subspaces of $\mathbb{F}_q^d$ and indeed there are $\frac{q^d-1}{q-1}$ such (on which the finite group $\mathbb{F}_{q^d}^\times/\mathbb{F}_q^\times$ acts transitively!) Now, for $i = 2, \ldots, d-1$, let $\Sigma_i = \{\gamma \in \Lambda \mid \tau((x_0, \gamma x_0)) = i\}$, i.e., the subset of Λ of those elements which takes x_0 to a neighbor of color i. As Λ acts simply transitively, $|\Sigma_i| = \left[\begin{smallmatrix} d \\ i \end{smallmatrix}\right]_q$ where $\left[\begin{smallmatrix} d \\ i \end{smallmatrix}\right]_q$ is the number of subspaces of $\mathbb{F}_q^d$ of codimension i. Let $\Sigma = \cup_{i=1}^{d-1} \Sigma_i$. One can deduce now that the 1-skeleton of $\mathcal{B}_d$ can be identified with $\mathrm{Cay}(\Lambda; \Sigma)$.

Now for every $0 \neq I \lhd \mathcal{O}_S$, we can define $\Lambda(I)$ as $\Lambda(I) = \ker(\Lambda \to \underline{G}(\mathcal{O}_S/I))$. This defines a complex $\Lambda(I)\backslash\mathcal{B}_d$ which by Theorem 2.3.1 is a Ramanujan complex.

Observe now that the building $\mathcal{B}$ is a clique complex, namely, a set of $i+1$ vertices forms a simplex if and only if every two vertices in it form a 1-edge. (In particular, the full structure of the complex is determined by the 1-skeleton.) The same is true for the quotients $\Lambda(I)\backslash\mathcal{B}_d$ (at least for large enough quotients, since the map $\mathcal{B}_d \to \Lambda(I)\backslash\mathcal{B}_d$ is a local isomorphism, moreover the injective radius of $\Lambda(I)\backslash\mathcal{B}_d$ grows logarithmically with respect to its size). So, these complexes are the Cayley complexes of the group $\Lambda/\Lambda(I)$ with respect to the set of generators Σ (or more precisely, its image in $\Lambda/\Lambda(I)$). Recall that a Cayley complex of a group H with respect to a symmetric set of generators Σ is the simplicial complex for which a subset Δ of H is a simplex if and only if $a^{-1}b \in \Sigma$ for every $a, b \in \Delta$. This is the clique complex determined by the Cayley graph $\mathrm{Cay}(H; \Sigma)$.

To make all this explicit also in the computer science sense, one needs to identify the quotients $\Lambda/\Lambda(I)$. This is carried out using the Strong Approximation Theorem. When I is a prime ideal of $\mathcal{O}_S$, we get that $\mathcal{O}_S/I$ is a finite field of order q^e for some e. The group $\Lambda/\Lambda(I)$ is then a subgroup of $\mathrm{PGL}_d(\mathbb{F}_{q^e})$ containing $\mathrm{PSL}_d(\mathbb{F}_{q^e})$. Various choices of ideals I can be made to make sure that any of the subgroups H between $\mathrm{PSL}_d(\mathbb{F}_{q^e})$ and $\mathrm{PGL}_d(\mathbb{F}_{q^e})$ can occur. Note that the quotient $\mathrm{PGL}_d(\mathbb{F}_{q^e})/\mathrm{PSL}_d(\mathbb{F}_{q^e})$ is a cyclic group of order dividing d. The resulting graphs are therefore t-partite for some $t \mid d$, just as in case $d = 2$ where we have had bi-partite and non-bipartite. We skip the technical details and give only a corollary (see Theorem 1.1 and Algorithm 9.2 in [LSV05b]).

Theorem 2.3.3. *Let q be a given prime power, $d \geq 2$ and $e \geq 1$. Assume $q^e \geq 4d^2$. Every subgroup H, with $\mathrm{PSL}_d(\mathbb{F}_{q^e}) \leq H \leq \mathrm{PGL}_d(\mathbb{F}_{q^e})$, has an (explicit) set Σ of $\left[\begin{smallmatrix} d \\ 1 \end{smallmatrix}\right]_q + \cdots + \left[\begin{smallmatrix} d \\ d-1 \end{smallmatrix}\right]_q$ generators, such that the Cayley complex of H with respect to Σ is a Ramanujan complex covered by $\mathscr{B}_d(F)$ when $F = \mathbb{F}_q((y))$.*

We mention in passing that the construction in this subsection is of interest even for $d = 2$ (in spite of the fact that we have already seen other constructions of Ramanujan graphs in the previous chapter) since for $d = 2$, $\mathbb{F}_{q^2}^\times / \mathbb{F}_q^\times$ acts transitively on *all* the $q + 1$ neighbors of the standard lattice. From this one can deduce that the resulting Ramanujan graphs are edge transitive and not merely vertex transitive (as it is always the case for Cayley graphs). This extra symmetry plays a crucial role in an application to the theory of error correcting codes (see [KL12]).

We hope that the higher Ramanujan complexes will also bear some fruits in combinatorics like their one dimensional counterparts. For first steps in this direction see [LM07, EGL14, FGL$^+$12]. For example in [LM07] and [EGL14] it is shown that they enjoy similar extremal properties as Ramanujan graphs: high girth and chromatic numbers (when these notions get the right interpretations for high dimensional complexes).

3. High dimensional expanders

In Definition 1.1.6 we presented the definition of expanding graphs. In recent years several suggestions have been proposed as to what should be the "right" definition of "expander" for higher dimensional simplicial complexes. In this chapter we will bring some of these as well as few results about the relations between them. This area is still in its primal state, and we can expect more developments. The importance of expanding graphs suggests that studying expanding simplicial complexes will also turn out to be very fruitful.

3.1. Simplicial complexes and cohomology

A finite simplicial complex X is a finite collection of subsets of a set $X^{(0)}$, called the set of vertices of X, which is closed under taking subsets. The sets in X are called *simplices* or *faces* and we denote by $X^{(i)}$ the set of simplices of X of dimension i, which are the sets of X of size $i + 1$. So $X^{(-1)}$ is comprised of the empty set, $X^{(0)}$—of the vertices, $X^{(1)}$—the edges, $X^{(2)}$—the triangles, etc. Throughout this discussion we will assume that $X^{(0)} = \{v_1, \ldots, v_n\}$ is the set of vertices and we fix an order $v_1 < v_2 < \cdots < v_n$ among the vertices. Now, if $F \in X^{(i)}$ we write $F = \{v_{j_0}, \ldots, v_{j_i}\}$ with $v_{j_0} < v_{j_1} < \cdots < v_{j_i}$. If $G \in X^{(i-1)}$, we denote the *oriented incidence number* $[F : G]$ by $(-1)^{\ell}$ if $F \backslash G = \{v_{j_\ell}\}$ and 0 if $G \nsubseteq F$. In particular, for every vertex $v \in X^{(0)}$ and for the unique face $\varnothing \in X^{(-1)}$, $[v : \varnothing] = 1$.

If $\mathbb{F}$ is a field then $C^i(X, \mathbb{F})$ is the $\mathbb{F}$-vector space of the functions from $X^{(i)}$ to $\mathbb{F}$. This is a vectors space of dimension $|X^{(i)}|$ over $\mathbb{F}$ where the characteristic functions $\{e_F \mid F \in X^{(i)}\}$ serve as a basis.

The coboundary map $\delta_i : C^i(X, \mathbb{F}) \to C^{i+1}(X, \mathbb{F})$ is given by:

$$(\delta_i f)(F) = \sum_{G \in X^{(i)}} [F : G] f(G).$$

So, if $f = e_G$ for some $G \in X^{(i)}$, $\delta_i e_G$ is a sum of all the simplices of dimension $i + 1$ containing G with signs ± 1 according to the relative orientations.

It is well known and easy to prove that $\delta_i \circ \delta_{i-1} = 0$. Thus $B^i(X, \mathbb{F}) = \operatorname{im} \delta_{i-1}$—"the space of i-coboundaries" is contained in $Z^i(X, \mathbb{F}) = \ker \delta_i$—the i-cocycles and the quotient $H^i(X, \mathbb{F}) = Z^i(X, \mathbb{F})/B^i(X, \mathbb{F})$ is the i-th cohomology group of X over $\mathbb{F}$.

In a dual way one can look at $C_i(X, \mathbb{F})$—the $\mathbb{F}$-vector space spanned by the simplices of dimension i. Let $\partial_i : C_i(X, \mathbb{F}) \to C_{i-1}(X, \mathbb{F})$ be the boundary map defined on the basis element F by: $\partial F = \sum_{G \in X^{(i-1)}} [F : G] \cdot G$, i.e., if $F = \{v_{j_0}, \ldots, v_{j_i}\}$ then $\partial_i F = \sum_{t=0}^{i} (-1)^t \{v_{j_0}, \ldots, \widehat{v_{j_t}}, \ldots, v_{j_i}\}$. Again $\partial_i \circ \partial_{i+1} = 0$ and so the boundaries $B_i(X, \mathbb{F}) = \operatorname{im} \partial_{i+1}$ are inside the cycles $Z_i(X, \mathbb{F}) = \ker \partial_i$ and $H_i(X, \mathbb{F}) = Z_i(X, \mathbb{F})/B_i(X, \mathbb{F})$ gives the i-th homology group of X over $\mathbb{F}$. As $\mathbb{F}$ is a field, it is not difficult in this case to show that $H_i(X, \mathbb{F}) \simeq H^i(X, \mathbb{F})$. Sometimes, it is convenient to identify $C_i(X, \mathbb{F})$ and $C^i(X, \mathbb{F})$ by assigning F to e_F.

The i-th Laplacian of X over F is defined as the linear operator $\Delta_i : C^i(X, \mathbb{F}) \to C^i(X, \mathbb{F})$ given by $\Delta_i = \partial_{i+1}\delta_i + \delta_{i-1}\partial_i$. The operator $\partial_{i+1}\delta_i$ is sometimes denoted (for clear reasons!) Δ_i^{up}, while $\Delta_i^{down} = \delta_{i-1}\partial_i$. In fact, ∂_{i+1} is the dual of δ_i and so the eigenvalues of Δ_i^{up} and Δ_{i+1}^{down} differ only by the multiplicity of zero. Note that what is customarily called the Laplacian of a graph is actually the upper 0-Laplacian:

$$\Delta_0^{up} f(x) = \deg(x) f(x) - \sum_{y \sim x} f(y).$$

3.2. $\mathbb{F}_2$-coboundary expansion

It seems that the first definition of higher dimensional expansion was given by Linial–Meshulam [LM06], Meshulam–Wallach [MW09] and Gromov [Gro10] (see also [DK12, GW12, SKM14, NR13]) as follows:

Definition 3.2.1. *For a simplicial complex X, the* $\mathbb{F}_2$-coboundary expansion of X in dimension i, $0 \le i < \dim X$, *is*

$$\mathcal{E}_i(X) = \min\Big\{ \frac{\|\delta_i f\|}{\|[f]\|} \;\Big|\; f \in C^i(X, \mathbb{F}_2) \backslash B^i(X, \mathbb{F}_2) \Big\}.$$

In other papers this notion is referred to as "cohomological expansion", "coboundary expansion", or "combinatorial expansion". Let us explain the notation here: $\mathbb{F}_2$ is the field of order two, for $f \in C^i$ (and similarly for $\delta f \in C^{i+1}$), $\|f\|$ is simply the number of i-simplices F for which $f(F) \neq 0$. Finally, $[f]$ is the coset $f + B^i(X, \mathbb{F}_2)$ and

$$\|[f]\| = \min\{\|g\| \mid g \in [f]\} = \min\{\|f + \delta_{i-1} h\| \mid h \in C^{i-1}(X, \mathbb{F}_2)\}.$$

One can see that $\|[f]\|$ is the minimal distance of f from $B^i(X, \mathbb{F}_2)$ in the Hamming metric, and in particular that $\|[f]\| = 0$ if and only if $f \in B^i(X, \mathbb{F}_2)$.

Let us explain why this artificially looking definition gives exactly expander graphs in the one dimensional case: If X is a graph, then $B^0 = \operatorname{im} \delta_{-1}$ is the one dimensional space containing two functions, the zero function 0 and the constant function $\mathbb{1}$ on all the vertices of X. Now, if $f \in C^0(X, \mathbb{F}_2)$ then f is nothing more than the characteristic function χ_A of some subset $A \subseteq X^{(0)}$, in which case $[f] = f + B^0(X, \mathbb{F}_2) = \{\chi_A, \chi_{\overline{A}}\}$ where $\overline{A}$ is the complement of A in $X^{(0)}$. Thus $\|[f]\| = \min(|A|, |\overline{A}|)$. Finally $\|\delta f\|$ is nothing more than the size of $E(A, \overline{A})$, i.e., the set of edges between A and $\overline{A}$. We can now see that the $\mathbb{F}_2$-coboundary expansion of X in dimension 0 (which is the only relevant dimension in this case) is exactly $\overline{h}(X)$ as in Remark 1.1.5.

Very few results have been proven so far about this concept. Here is one of them (see [MW09, Gro10]):

Proposition 3.2.2. *The complete complex $\Delta_{[n-1]}$, the simplicial complex on n vertices where every subset is a face, has $\mathbb{F}_2$-coboundary expansion $\frac{n}{i+1}$ at dimension i, $1 \le i \le n-1$.*

Remark 3.2.3. One should note that X has positive $\mathbb{F}_2$-coboundary expansion in dimension i if and only if $H^i(X, \mathbb{F}_2) = 0$: If $Z^i(X, \mathbb{F}_2) = B^i(X, \mathbb{F}_2)$ then $\delta f \neq 0$ for every $f \in C^i \backslash B^i$, while if $f \in Z^i(X, \mathbb{F}_2) \backslash B^i(X, \mathbb{F}_2)$ then $\delta f = 0$ and $\|[f]\| \neq 0$. This vanishing of $H^{d-1}(X, \mathbb{F}_2)$ in the graph case, $d = 1$, is the vanishing of $H^0(X, \mathbb{F}_2)$ which means that the graph X is connected. Indeed, it is clear that an ε-expander graph is connected.

Most of the known results on coboundary expansion refer to complexes X of dimension d whose $(d-1)$-skeleton is complete (i.e., every subset of $X^{(0)}$ of size d is a face in the complex). See [LM06, MW09, Gro10, DK12, Wag11, GW12, LM13] for various results, mainly on random complexes. There are also some other complexes, e.g. finite spherical buildings ([Gro10, LMM14]) and base-transitive matroidal complexes ([LMM14]) which were proved to be coboundary expanders, but all of them are of unbounded degree (see [LM13] for bounded edge degree). Note that sometimes (especially in the unbounded case) one prefers to use a variant of the above definition and to give weights to the cells according to the number of faces containing them—see [LMM14] for example.

As far as we know there is no known family of higher dimensional $\mathbb{Z}_2$-coboundary expanders of *bounded* degree (i.e., the number of faces containing a vertex is bounded). It is natural to suggest that the Ramanujan complexes of Chap. 2 (and even more generally, all finite quotients of higher dimensional Bruhat–Tits buildings of simple groups of rank ≥ 2 over local fields) are such. But this is not the case in general. For example, let Γ be any cocompact lattice in $\mathrm{PGL}_3(F)$ where F is a local field and assume $\Gamma/[\Gamma,\Gamma]\Gamma^2$ is non-trivial (i.e., Γ has a non-trivial abelian quotient of 2-power order—by [Lub87] every lattice has such a sublattice of finite index) then $H^1(\Gamma\backslash\mathscr{B},\mathbb{F}_2) \neq 0$ (since $\mathscr{B}$—the Bruhat–Tits building of $\mathrm{PGL}_3(F)$ is contractible) and so by Remark 3.2.3, the $\mathbb{F}_2$-coboundary expansion of $X = \Gamma\backslash\mathscr{B}$ in dimension 1 is 0. It might be that the vanishing of the cohomology is the only obstruction.

Another possible way to circumvent this is to use instead the notion of Gromov of "cofilling": The cofilling of X (in dimension i) is

$$\nu_i(X) = \max\left\{\frac{\|f + Z^i\|}{\|\delta_i f\|} \;\middle|\; f \in C^i(X,\mathbb{F}_2)\backslash Z^i(X,\mathbb{F}_2)\right\}.$$

When $H^i(X,\mathbb{F}_2)$ vanishes, the cofilling and the $\mathbb{F}_2$-coboundary expansion are related by $\nu_i(X) = \frac{1}{\mathscr{E}_i(X)} = [\min_{f\in C^i\backslash B^i} \frac{\|\delta_i f\|}{\|f+B^i\|}]^{-1}$. When $H^i(X,\mathbb{F}_2)$ does not vanish, $\mathscr{E}_i(X)$ is zero (see Remark 3.2.3), but $\nu_i(x)$ is always finite since $\|\delta_i f\| \neq 0$ for $f \notin Z^i$. For example, the Cheeger constant $\overline{h}$ vanishes for a disconnected graph, while $\frac{1}{\nu_0(x)}$ is the *mediant* (or "freshman sum") of the Cheeger constants of the connected components of the graph, and it is always positive. We present the following conjecture:

Conjecture 3.2.4. Let $\mathscr{B}$ be the Bruhat–Tits building associated with $\mathrm{PGL}_d(F)$, F a local field and $d \geq 3$. There exists a constant $\nu = \nu(d,F)$ such that $\nu_i(X) \leq \nu$ for every finite quotient X of $\mathscr{B}$ and $i < d$.

For some very special cases of this conjecture see [KKL14]. For some applications of coboundary expansions to computer science—see [NR13] (computational geometry) and [KL14] (property testing).

3.3. The Cheeger constant

The Cheeger constant $h(X)$ for a graph X is defined in Definition 1.1.4 above (see also Remark 1.1.5 there). One may argue what should be the right definition of $h(X)$ when X is a higher dimensional simplicial complex. The following definition is given in [PRT12]:

Definition 3.3.1. *For a d-dimensional simplicial complex X, denote*

$$h(X) = \min_{X^{(0)}=\coprod_{i=0}^{d} A_i} \frac{|X^{(0)}||F(A_0,\dots,A_d)|}{|A_0|\cdot \dots \cdot |A_d|},$$

where the minimum is over all the partitions of $X^{(0)}$ into non-empty sets $A_0,\dots,A_d$ and $F(A_0,\dots,A_d)$ denotes the set of d-dimensional simplices with exactly one vertex in each A_i.

For $d = 1$, it coincides with Definition 1.1.4. But, in a way, this definition keeps the spirit of the mixing lemma (Proposition 1.1.8): $h(X)$ measures the number of "edges" (i.e., d-faces) "between" (i.e., with single representatives in each of the) A_i. The quantity $|F(A_0,\dots,A_d)|$ is "normalized" by multiplying it by $\frac{|X^{(0)}|}{\prod_{i=0}^{d}|A_i|}$.

This definition works well when X has a complete $(d-1)$-skeleton (see more in §3.5), but it gives zero whenever $X^{(d-1)}$ is not complete (If $G = \{v_0,\dots,v_{d-1}\} \notin X^{(d-1)}$ take $A_i = \{v_i\}$ for $i = 0,\dots,d-1$ and $A_d = X^{(0)}\setminus G$. Then $F(A_0,\dots,A_d) = \varnothing$). This calls for a modified definition which will be interesting for complexes with non-complete skeleton. One such modification is suggested in [PRT12]:

$$\widetilde{h}(X) = \min_{X^{(0)}=\coprod_{i=0}^{d} A_i} \frac{|X^{(0)}||F(A_0,\dots,A_d)|}{|F^{\partial}(A_0,\dots,A_d)|}$$

where $F^{\partial}(A_0,\dots,A_d)$ is the set of "$(d-1)$-spheres" (namely, copies of the $(d-1)$-skeleton of the d-simplex) which have one vertex in each A_i. For a complex with a complete skeleton, $h(X)$ and $\widetilde{h}(X)$ coincide.

While $\widetilde{h}(X)$ is of interest for some complexes with incomplete skeleton, for clique complexes (such as the Ramanujan complexes constructed in Chap. 2) it is trivial, since every "sphere" in $F^{\partial}(A_0,\dots,A_d)$ bounds a d-simplex, giving immediately $\widetilde{h}(X) = |X^{(0)}|$. In [GP14] another Cheeger-type constant is suggested, which is interesting for clique complexes as well: fixing some $0 \le \alpha < 1$, we define

$$h_\alpha(X) = \min\left\{ \frac{|X^{(0)}||F(A_0,\dots,A_d)|}{|A_0|\cdot \dots \cdot |A_d|} \;\middle|\; \begin{array}{l} X^{(0)} = \coprod_{i=0}^{d} A_i, \\ \forall i\ |A_i| > \alpha|X^{(0)}| \end{array} \right\}.$$

This constant coincides with $h(X)$ for $\alpha = 0$, but for positive α it avoids unbalanced partitions, which have $F(A_0, \dots, A_d) = \varnothing$ trivially for complexes with bounded degrees and $|X^{(0)}|$ large enough.

Another natural challenge, in the spirit of the Cheeger constant, is to bound $|F(A_0, \dots, A_d)|$ for sets A_i which do not form a partition. In [PRT12], the authors call the difference

$$\left| |F(A_0, \dots, A_d)| - \frac{|X^{(d)}||A_0| \cdot \dots \cdot |A_d|}{\binom{n}{d+1}} \right| \tag{3.1}$$

the *discrepancy* of $A_0, \dots, A_d$. It turns out that the discrepancy, as well as the various generalizations of the Cheeger constant can be bounded in terms of the spectrum of the Laplacian. This brings us to our next subject.

3.4. Spectral gap

In Chap. 1 we saw that the notion of expander can be described by means of the eigenvalues of the adjacency matrix A of the graph. For a k-regular graph X, the matrix A is nothing more than $A = kI - \Delta_0^{up}$ where Δ_0^{up} is the 0-dimensional upper Laplacian of X over $\mathbb{F} = \mathbb{R}$ as defined in §3.1. We can translate Theorem 1.1.7 to deduce that a family of k-regular graphs $\{X_t\}_{t \in I}$ is a family of expanders if and only if there exists $\varepsilon > 0$ such that every eigenvalue λ of $\Delta_0^{up}|_{Z_0(X,\mathbb{R})} = \Delta_0|_{Z_0(X,\mathbb{R})}$ satisfies $\lambda \geq \varepsilon$ (the last is equality is since $\Delta_i^{down}(Z_i(X, \mathbb{R})) = \delta_{i-1}\partial_i(\ker \partial_i) = 0$). Note that $Z_0(X, \mathbb{R}) = \{f : X^{(0)} \to \mathbb{R} \mid \sum_{x \in X^{(0)}} f(x) = 0\}$. It is therefore natural to generalize and to define:

Definition 3.4.1. *Let X be a simplicial complex of dimension d and $0 \leq i \leq d-1$. We denote $\lambda_i(X) = \min \operatorname{Spec}(\Delta_i|_{Z_i(X,\mathbb{R})})$ and we say that X has spectral gap $\lambda_i(X)$ in dimension i. We write $\lambda(X)$ for $\lambda_{d-1}(X)$.*

It is natural to expect that just like in graphs where there is a direct connection between the Cheeger constant and the spectral gap, something like that should happen in the higher dimensional case, but examples presented in [PRT12] show that there exist simplicial complexes with $\lambda(X) = 0$ while $h(X) > 0$. Nevertheless, spectral expansion does imply a Cheeger bound:

Theorem 3.4.2 ([PRT12]). *For a finite complex with a complete skeleton, $\lambda(X) \leq h(X)$.*

A similar generalization is obtained in [PRT12] for the expander mixing lemma (Proposition 1.1.8 above). Given any two sets of vertices $A, B \subseteq V$, the mixing lemma for graphs bounds the deviation of $|E(A, B)|$ from its expected value in a random k-regular graph, in terms of the spectral invariant μ_0. From the perspective of the simplicial Laplacian, μ_0 is the spectral radius of $kI -$

$\Delta_0|_{Z_0(X,\mathbb{R})}$, i.e., the maximal absolute value of its eigenvalues. The following generalization then holds for higher dimensional complexes:

Theorem 3.4.3 ([PRT12]). *Let X be a finite d-dimensional complex with a complete $(d-1)$-skeleton. Let k be the average degree of a $(d-1)$-cell, and define*

$$\mu_0(X) = \max\{|\gamma| \mid \gamma \in \mathrm{Spec}(kI - \Delta_{d-1}|_{Z_{d-1}(X,\mathbb{R})})\}.$$

Then for every disjoint sets of vertices $A_0, \dots, A_d$,

$$\left| |F(A_0, \dots, A_d)| - \frac{k|A_0| \cdot \dots \cdot |A_d|}{|X^{(0)}|} \right| \le \mu_0(X)(|A_0| \cdot \dots \cdot |A_d|)^{\frac{d}{d+1}}.$$

When specializing to $d = 1$ this gives the original expander mixing lemma for graphs, except for the additional assumption that the sets of vertices are disjoint. The reader is referred to [PRT12] for the proofs of Theorems 3.4.2 and 3.4.3.

For complexes with non-complete skeleton, it was conjectured in [PRT12] that Theorem 3.4.2 should hold for the constant $\widetilde{h}(X)$. This was recently shown to be true:

Theorem 3.4.4 ([GS14]). *For any finite complex, $\lambda(X) \le \widetilde{h}(X)$.*

The second variant of the Cheeger constant, $h_\alpha(X)$, is bounded in [GP14] for triangle complexes. The bound involves not only $\lambda(X) = \lambda_1(X)$, but also the spectrum of the vertex Laplacian:

Theorem 3.4.5 ([GP14]). *If X is a triangle complex, and for some k the spectral radius of $kI - \Delta_0|_{Z_0(X,\mathbb{R})}$ is μ_0, then*

$$h_\alpha(X) \ge \frac{\lambda_1}{|X^{(0)}|}\Big(k - \mu_0\Big(1 + \frac{10}{9\alpha^3}\Big)\Big).$$

Note that this implies Theorem 3.4.2 (for triangle complexes), as the complete graph has $\mu_0 = 0$ for $k = |X^{(0)}|$.

In the same spirit, an expander mixing lemma for arbitrary complexes is proved in [Par13], using the spectra of all Laplacians together:

Theorem 3.4.6 ([Par13]). *Let X be a complex of dimension d, such that*

$$\mathrm{Spec}(\Delta_i|_{Z_i(X,\mathbb{R})}) \subseteq [k_i - \mu_i, k_i + \mu_i]$$

for $0 \le i \le d-1$. Then for any disjoint sets of vertices $A_0, \dots, A_d$,

$$\begin{aligned} &\left| |F(A_0, \dots, A_d)| - \frac{k_0 \cdots k_{d-1}}{n^d} |A_0| \cdot \dots \cdot |A_d| \right| \\ \le & c_d k_0 \cdots k_{d-1} \Big(\frac{\mu_0}{k_0} + \dots + \frac{\mu_{d-1}}{k_{d-1}} \Big) \max |A_i| \end{aligned}$$

where c_d depends only on d.

For an "expander mixing lemma" for Ramanujan complexes—see [EGL14] —where it is applied to give a lower bound on their chromatic number.

It is natural to suggest some extension of Alon–Boppana theorem (Theorem 1.1.2) to this high dimensional case (see also Theorem 2.1.4). In [PR12] it is shown that the high dimensional analogue of Alon–Boppana indeed holds in several interesting cases (for example, for quotients of an infinite complex with non-zero spectral gap), but that it can also fail. For a "general" Alon–Boppana type theorem—see [Fi14].

The most important work so far on the spectral gap of complexes is the seminal work of Garland [Gar73]. As this work has been described in many placed (e.g. [Bor73, Zuk96, GW12]) we will not elaborate on it here. We just mention that Garland proved Serre's conjecture that $H^i(X,\mathbb{R}) = 0$ for every $1 \leq i \leq d-1$ where X is a finite quotient of the Bruhat–Tits building of a simple group of rank $d \geq 2$ over a local field $\mathbb{F}$. He did this by proving a bound on the spectral gaps which depends only d and $\mathbb{F}$ (the i-th cohomology group over $\mathbb{R}$ vanishes if and only if the corresponding spectral gap λ_i is non-zero).

It is still not clear what is the relation between the coboundary expansion and the spectral gap. See [GW12, SKM14] where some complexes are presented with $\lambda_i(X)$ arbitrarily small while $\mathscr{E}_i(X)$ is bounded away from zero, and the other way around.

3.5. The overlap property

An interesting "overlap" property for complexes, which is closely related to expanders, was defined by Gromov [Gro10], and was further studied in [FGL$^+$12, MW14, Kar12, KKL14, KW]. We need first some notation: Let X be a d-dimensional simplicial complex and $\varphi : X^{(0)} \to \mathbb{R}^d$ an injective map. The map φ can be extended uniquely to a simplicial mapping $\widetilde{\varphi}$ from X (considered now as a topological space in the obvious way) to $\mathbb{R}^d$ (i.e., by extending φ affinely to the edges, triangles, etc.) This will be called a geometric extension. The map φ can be extended in many different ways to a continuous map $\widetilde{\varphi}$ from the topological simplicial complex X to $\mathbb{R}^d$, such $\widetilde{\varphi}$ will be called topological extensions.

Definition 3.5.1. *Let X be a d-dimensional simplicial complex and* $0 < \varepsilon \in \mathbb{R}$. *We say that X has* ε-geometric overlap *(resp.* ε-topological overlap*) if for every injective map $\varphi : X^{(0)} \to \mathbb{R}^d$ and a geometric (resp. topological) extension $\widetilde{\varphi} : X \to \mathbb{R}^d$, there exists a point $z \in \mathbb{R}^d$ such that $\widetilde{\varphi}^{-1}(z)$ intersects at least $\varepsilon \cdot |X^{(d)}|$ of the d-dimensional simplices of X.*

To digest this definition, let us spell out what does it mean for expander graphs: Let $\varphi : X^{(0)} \to \mathbb{R}$ be an injective map and $\widetilde{\varphi}$ any continuous extension of it to the graph. Let $z \in \mathbb{R}$ be a point such that $\lfloor \frac{1}{2}|X^{(0)}| \rfloor$ of the images of the vertices are above it (and call $L \subseteq X^{(0)}$ this set of vertices) and the rest are below it. Then $\widetilde{\varphi}^{-1}(z)$ intersects all the edges of $E(L, \overline{L})$ (= the set of edges going from L to its complement). If X is an ε-expander k-regular graph, then $X^{(1)} = \frac{|X^{(0)}|k}{2}$ while $|E(L, \overline{L})| \geq \frac{\varepsilon}{2}|L| \approx \frac{\varepsilon}{2}\frac{|X^{(0)}|}{2} = \frac{\varepsilon}{2k}|X^{(1)}|$. Thus X has the $\frac{\varepsilon}{2k}$-topological overlapping property.

The reader should notice however that this property is *not* equivalent to expander. In fact, it does not even imply that the graph X is connected. It can be a union of a large expanding graph and a small connected component. Still, this property captures the nature of expansion especially in the higher dimensional case.

It is interesting to mention that while it is trivial to prove that the complete graph is an expander, it is a non-trivial result that the higher dimensional complete complexes have the overlap property. This was proved for the geometric overlap in [BF84] for dim 2 and in [Bár82] for all dimensions. For the topological overlap, this was proved in [Gro10] (see also [MW14, Kar12, KW]).

The main result of [FGL$^+$12] asserts that there even exist simplicial complexes with the geometric overlapping property of bounded degree. They prove it by several methods: probabilistic and constructive. The constructive examples are the Ramanujan complexes which were discussed in length in Chap. 2 (but under the assumption that q is large enough with respect to d). In fact, the proof there is valid for all the finite quotients of $\mathscr{B} = \mathscr{B}(\mathrm{PGL}_d(\mathbb{F}))$ and not only to the Ramanujan ones (again assuming $q \gg d$). It is quite likely that the same result holds also for the other Bruhat–Tits buildings of simple groups of rank ≥ 2 [Ev14]*. In all these results the following theorem of Pach plays a crucial role:

Theorem 3.5.2 ([Pac98]). *For every $d \geq 1$, there exists $c_d > 0$ such that for every $d+1$ disjoint subsets $P_1, \dots, P_{d+1}$ of n points in general position in $\mathbb{R}^d$, there exists $z \in \mathbb{R}^d$ and subsets $Q_i \subseteq P_i$ with $|Q_i| \geq c_d|P_i|$ such that every d-dimensional simplex with exactly one vertex in each Q_i, contains z.*

Let us show now, following [PRT12] how to deduce the geometric overlap property from Pach's theorem and the mixing lemma, when we have a "concentration of the spectrum". Let X be a d-dimensional complex on n vertices, with a complete $(d-1)$-skeleton. For an arbitrary injective map $\varphi : X^{(0)} \to \mathbb{R}^d$ we can divide $\varphi(X^{(0)})$ to $(d+1)$-disjoint sets $P_0, \dots, P_d$, each of order (approximately) $\frac{n}{d+1}$. By Pach's theorem there is a point $z \in \mathbb{R}^d$ and subsets $Q_i \subseteq P_i$ of sizes $|Q_i| = \frac{c_d n}{d+1}$, such that z belongs to every d-simplex formed by representatives from $Q_0, \dots, Q_d$. This means that for the geometric extension $\widetilde{\varphi} : X \to \mathbb{R}^d$, $\widetilde{\varphi}^{-1}(z)$ intersects every simplex in $F(\varphi^{-1}(Q_0), \dots, \varphi^{-1}(Q_d))$.

* See [FGL$^+$12] for another constructive method based on Ramanujan graphs and for a random construction.

Turning to the mixing lemma (Theorem 3.4.3 above), if the average degree of a $(d-1)$-cell in X is k, and $\operatorname{Spec} \Delta_{d-1}|_{Z_{d-1}(X,\mathbb{R})} \subseteq [k-\varepsilon, k+\varepsilon]$, then

$$\begin{aligned} |F(\varphi^{-1}(Q_0),\ldots,\varphi^{-1}(Q_d))| &\geq \frac{k|Q_0|\cdots|Q_d|}{n} - \varepsilon(|Q_0|\cdots|Q_d|)^{\frac{d}{d+1}} \\ &= \Big(\frac{c_d n}{d+1}\Big)^d \Big(\frac{kc_d}{d+1} - \varepsilon\Big). \end{aligned}$$

Since this applies to every $\varphi : X^{(0)} \to \mathbb{R}^d$, the quotient by $|X^d| = \frac{k}{d+1}\binom{n}{d}$ gives a lower bound for the geometric expansion of X:

$$\operatorname{overlap}(X) \geq \frac{(\frac{c_d n}{d+1})^d(\frac{kc_d}{d+1} - \varepsilon)}{|X^d|} \geq \frac{c_d^d}{e^{d+1}}\Big(c_d - \frac{\varepsilon(d+1)}{k}\Big).$$

While bounds on the spectrum give some geometric overlap properties, it is much more difficult to get the topological overlap property. The only method known to us is via the following Theorem of Gromov (see [KW] for a simplified proof; though still highly non-trivial):

Theorem 3.5.3. *If X has $\mathbb{F}_2$-coboundary expansion $\varepsilon_i(X) \geq \eta\frac{|X(i+1)|}{|X(i)|}$ for all $0 \leq i \leq d-1$, then X has the ε-topological overlap property for some $\varepsilon = \varepsilon(\eta, d) > 0$.*

Still, we do not know any example of higher dimensional complexes of bounded degree with the $\mathbb{F}_2$-coboundary expansion property. It is tempting to conjecture that the finite quotients X of a fixed high-rank Bruhat–Tits building of dimension d, with trivial cohomology over $\mathbb{F}_2$, form such a family. As mentioned above they were proved in [FGL+12] to have the geometric overlapping property. But as observed in §3.2, they are not coboundary expanders as the $\mathbb{F}_2$-cohomology does not necessarily vanish. Kaufman noticed that Theorem 3.5.3 can be strengthened to include cases with non-trivial cohomology provided any non-trivial cocycle satisfies a linear systolic lower bound (for a complete and simplified proof of Theorem 3.5.3 and its stronger form, see [KW]). This extension enabled Kazhdan, Kaufman and Lubotzky [KKL14] to prove the existence of 2-dimensional simplicial complexes with bounded (vertex) degree and with the topological overlapping property. These are the 2-skeletons of Ramanujan complexes of dimension 3 covered by the Bruhat–Tits building of $\mathrm{PGL}_4(\mathbb{F}_q((t)))$, when q is sufficiently large (and fixed). The problem for higher d and for the Ramanujan complexes themselves (even for dimension 2) is still open.

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